On circular Cylinders by Four or Five Points in Space

Olivier Devillers, Bernard Mourrain, Franco P. Preparata, Philippe Trebuchet

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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On circular cylinders by four or five points in space

Olivier Devillers[*] , Bernard Mourrain[†] , Franco P. Preparata[‡] , Philippe Trebuchet[§]

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Projets Prisme et Saga

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Abstract: We are interested in computing effectively cylinders through 5 points, and in other problems involved in metrology. In particular, we consider the cylinders through 4 points with a fix radius and with extremal radius. For these different problems, we give bounds on the number of solutions and exemples show that these bounds are optimal. Finally, we describe two algebraic methods which can be used here to solve efficiently these problems and some experimentation results.

Key-words: Geometry, algebra, resolution, cylinder, metrology

[*] INRIA, BP93, 06902 Sophia-Antipolis, France, Olivier.Devillers@sophia.inria.fr
[†] INRIA, BP93, 06902 Sophia-Antipolis, France, Bernard.Mourrain@sophia.inria.fr
[‡] Center for Geometric Computing, Department of Computer Science, Brown University,115 Waterman Street, Providence, RI 02912. franco@cs.brown.edu
[§] INRIA, BP93, 06902 Sophia-Antipolis, France, Philippe.Trebuchet@sophia.inria.fr

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, B.P. 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : 04 92 38 77 77 - International : +33 4 92 38 77 77 —Fax : 04 92 38 77 65 - International : +33 4 92 38 77 65

A propos de cylindres passant par 4 et 5 points

Résumé : Nous nous intéressons au calcul de cylindres passant par 5 points, ainsi qu'a des problèmes similaires apparaissant par exemple en métrologie. Nous considérons en particulier les cylindres par 4 points et de rayon fixés, ainsi que les cylindres par 4 points et de rayon extremal. Pour ces différents problèmes, nous donnons des bornes sur le nombre de solutions et des exemples montrant que ces bornes sont optimales. Enfin, nous décrivons deux méthodes algébriques permettant de résoudre effectivement ces problèmes ainsi que quelques résultats d'expérimentations.

Mots-clés : Géométrie, algèbre, résolution, cylindre, métrologie

1 Introduction

The focus of this paper is the analysis of circular cylinders through sets of points in three dimensions. This investigation has a variety of motivations. Clearly, if a cylinder of radius R and direction $\mathbf{t}$ passes through a set of points P, it means also that there is a line of direction $\mathbf{t}$ tangent to the spheres of radius R and centred at the points of P. Another interpretation is that an observer looking at P from infinity in direction $\mathbf{t}$ sees the points of P as cocircular. So, this kind of operations arises in surface reconstruction, visibility, and metrology.

The paper proves the following results. We first describe precisely, in Section 2, some results on cylinders through points in special configurations, specifically, for the vertices of one regular tetrahedron or of two regular tetrahedra, either disjoint or sharing a face. These preliminary observations help us to visualize the problem and provide lower bounds on the number of solutions.

In Section 3, given four points in space, we prove a degree-3 conditions on the direction of a cylinder through these four points. We exploit this result to prove that there are at most six cylinders through five points (Section 4), at most 186 cylindrical shells through six points (Section 5), and at most nine pairs of parallel cylinders through two sets of four points (Section 6). These bounds are tight as illustrated by explicit examples. Such primitive are used for smallest enclosing cylinders [1, 17] or for Delaunay triangulation of projected points [4]. The number of cylindrical shells appear in metrology problems [2, 8].

The first problem of cylinders through 5 points has already been considered in the literature. The problem of cylinders through 4 and 5 points is discussed by Bottema and VeldKamp [5], using "complicated" line geometry; in particular, they gave a bound on the number of cylinders through 5 points. This problem has been reinvestigated by Lichtblau [12] using calculus with a computer algebra system.

Searching for cylinders of some fixed radius can be interesting for collision detection, visibility or the exist for the existence of enclosing cylinders with a given radius. We investigate this problem in Section 7 and prove a tight bound of 12 solutions. The same bound was recently obtained by Mac Donald, Pach and Theobald [9] using a different approach, and its tightness is established also using the regular tetrahedron.

The smallest (or largest) cylinder enclosing (or "surrounded by") a set of points can be defined by 5 or fewer points, so that it is of interest to study extremal-radius cylinders through 4 points. We prove in Section 8 that their number is at most 18.

2 On the views of regular tetrahedra

2.1 A single regular tetrahedron

In this section, we will investigate the different views of the four vertices of a regular tetrahedron. More precisely given 4 points $p = (0, 0, 0)$ (pink), $b = (1, 0, 0)$ (blue), $r = (1/2, \sqrt{3}/2, 0)$ (red) and $o = (1/2, \sqrt{3}/6, \sqrt{6}/3)$ (orange), we will project these four points on a plane orthogonal to the direction $\mathbf{t}$ and look at the Delaunay triangulation of the four points. By definition, the direction $\mathbf{t}$ is an element of the projective plane $\mathbb{P}^2$. It will be represented by 3 coordinates $\mathbf{t} = (l : m : n)$ with $(l, m, n) \neq \mathbf{0}$ and $(l : m : n) \equiv \lambda(l : m : n)$ for $\lambda \neq 0$. Conveniently, $\mathbf{t}$ can also be viewed as a unit vector, and we identify $\mathbf{t}$ with $-\mathbf{t}$. We draw on the sphere of directions a diagram $\mathcal{D}$ corresponding to the different Delaunay triangulations of the four points viewed along direction $\mathbf{t}$. Since $\mathbf{t}$ and $-\mathbf{t}$ define the same direction, only half of the sphere is relevant. Thus we are seeking directions where the topology of the Delaunay triangulation changes.

Clearly, if $\mathbf{t}$ is parallel to a face of the tetrahedron, then three points are collinear in projection. In such a case $\mathbf{t}$ describes a point of an edge of $\mathcal{D}$, separating a region where the convex hull of the projected point is a triangle from one where it is a quadrilateral; these directions yield 4 great circles of the unit sphere (belonging to $\mathcal{D}$), drawn in blue in Figure 1.

If $\mathbf{t}$ is parallel to the plane generated by two opposite edges of the tetrahedron, the four projected points form a trapezoid with an axis of symmetry, so that the four points are clearly cocircular; thus we get three other great circles, drawn in orange in Figure 1, belonging to $\mathcal{D}$. We point out that for four points in general position, the orange curve no longer consists of a set of great circles, but as we will see in Section 3, it will be a general cubic curve.

Figure 1 shows a view from above of the diagram $\mathcal{D}$ on the sphere of directions. Orange curves correspond to cocircularity directions and blue curves to collinearity directions. The intersections between blue curves correspond to the six directions of the edges of the tetrahedron. In such a case, two vertices project at the same point, so that the Delaunay triangulation is a single triangle, and thus the four points are cocircular. It follows that each of these directions belongs to a branch of the orange curve. These points are marked by a triangle on Figure 1. Figure 2 illustrates a projective version of Figure 1.

At the intersection of two branches of the orange curve, the projection of the four points is a square, and these three directions are marked by a square. When the direction of projection moves from a *triangle*-direction to a *square*-direction on the orange curve the radius of the circle circumscribing the four points is decreasing, so that the circle passing through the four points has minimal radius ($\frac{1}{2} = 0.5$) at the three *square*-directions, maximal radius ($\frac{3\sqrt{2}}{8} \simeq 0.53$) at the six *triangle*-directions and any given radius in between in at least twelve different points (one on each orange edge between a triangle-point and a square-point). This will be confirmed by the algebraic developments of Section 7. In Section 8, we will analyze in detail the multiplicity of these extremal points.

2.2 Two regular tetrahedra sharing a face

Referring to Figure 3, if we add the point $g = (1/2, -\sqrt{3}/6, -\sqrt{6}/3)$ (green) and consider the two tetrahedra (p, b, r, o) and (p, b, r, g), we can draw on the sphere the two corresponding diagrams. Due to the symmetry of the configuration, the diagram of (p, b, r, g) is obtained from the diagram of (p, b, r, o) by a rotation of π.

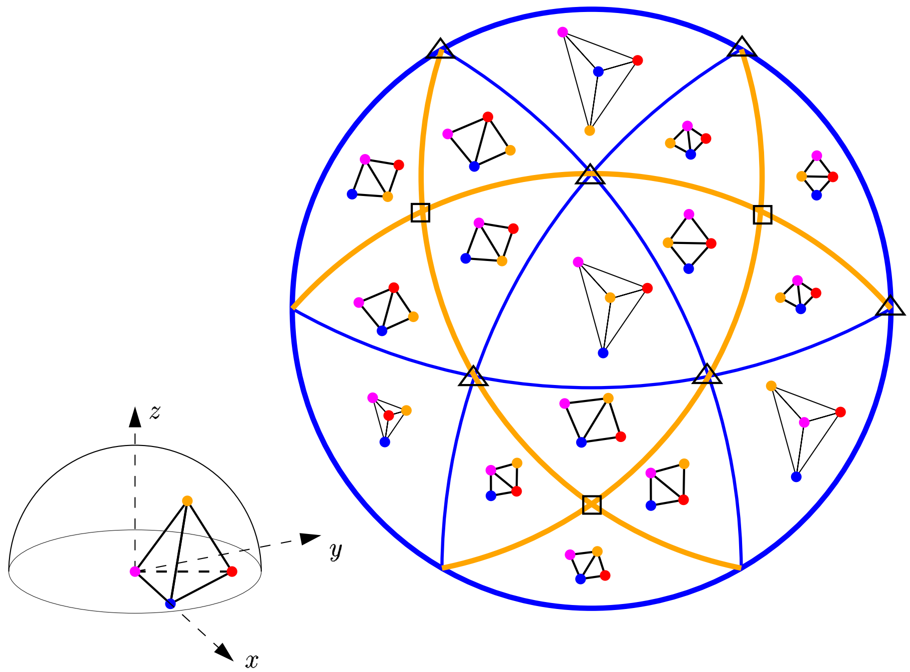

Figure 1: Different views of fours points.

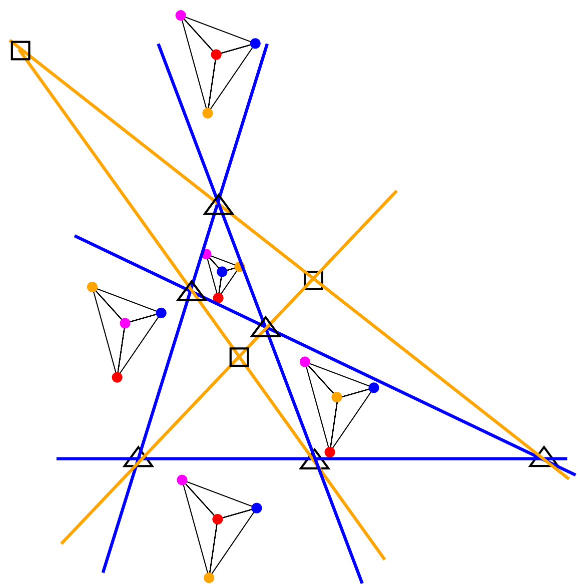

Figure 2: The same picture in the projective plane.

In Figure 3 collinearity curves are in blue for (p, b, r, o) and in purple for (p, b, r, g), cocircularity curves are in orange for (p, b, r, o) and in green for (p, b, r, g). There are six directions along which the five points have cocircular projections, marked by a circle on the figure, at the intersection of orange and green cocircularity curves. There are also three directions where two of the points $(p, b, r,)$ project on the same point, and thus (p, b, r, g) and (p, b, r, o) are cocircular but on different circles; these points are marked by a pair of intersecting circles in the figure.

Thus for the six directions where the five points project cocircularly, there is a circular cylinder with that direction passing through the five points.

2.3 Two disjoint regular tetrahedra

Given two sets of four points, we are interested in the directions along which both sets have cocircular projections. As noted in Section 2.1 the set of directions along which four points have cocircular projections is defined by three great circles on the sphere. Thus given two regular tetrahedra we get nine directions belonging to the corresponding two sets of three great circles.

Figure 4 describes a case where both regular tetrahedra have a horizontal face, so that the diagram for the second tetrahedron is just a rotation of the first one. Each of the nine directions is marked by two small circles on the figure.

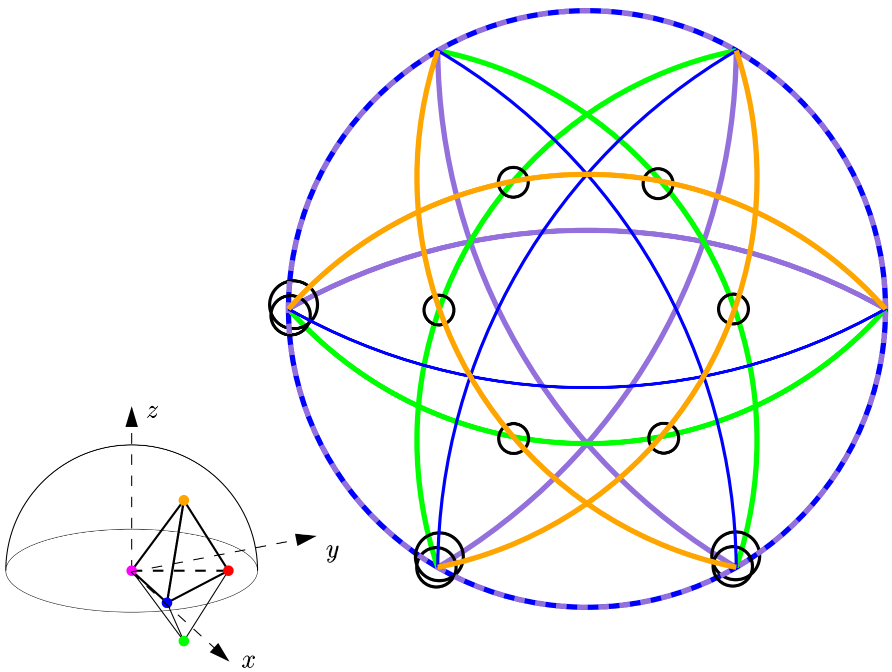

Figure 3: There are 6 cylinders through five points.

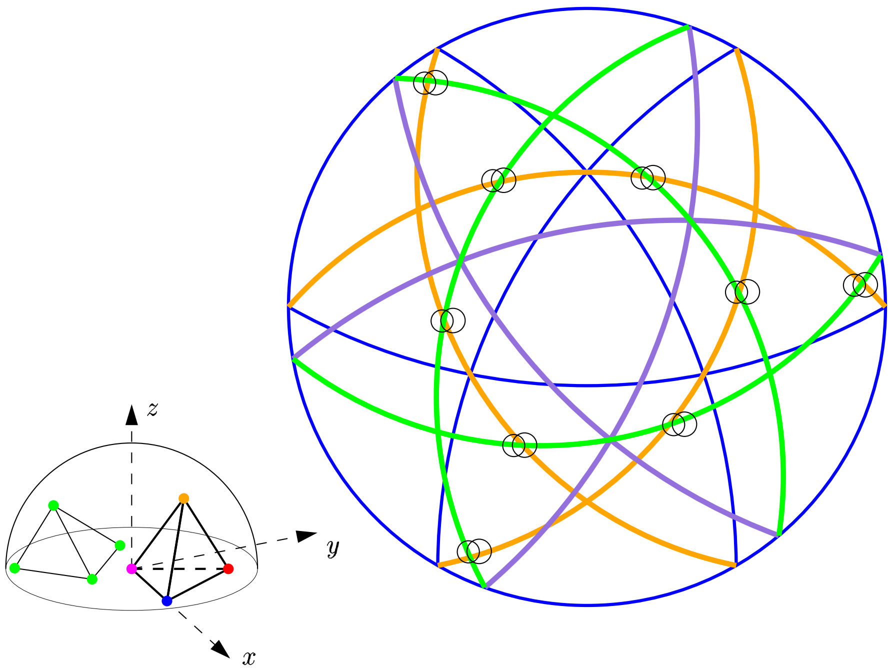

Figure 4: There are 9 pairs of parallel cylinders through two sets of four points.

3 Cylinders through four points

With the intuition supplied by the preceding geometric considerations, we can now move to an algebraic analysis. We begin by considering the equations defining the set of cylinders passing by four points in the space. Given four points p_1, p_2, p_3, p_4 in three-dimensional space, we wish to analyze the set of directions corresponding to circular cylinders passing by these points. Without loss of generality, modulo a rigid motion, we may assume that $p_1 = (0,0,0)$, $p_2 = (x_2, 0, 0)$, $p_3 = (x_3, y_3, 0)$ and $p_4 = (x_4, y_4, z_4)$.

Let $\mathbf{t} = (l, m, n)$, with $l^2 + m^2 + n^2 = 1$, be the unit vector identifying a direction ($\mathbf{t} \equiv (l : m : n)$ in $\mathbb{P}^2$). Consider now the plane π through the origin and orthogonal to $\mathbf{t}$, and a system of coordinates (X, Y, Z) having its two first axes in π and the third axis of direction $\mathbf{t}$. The transformation of coordinates from (x, y, z) to (X, Y, Z) brings the Z-axis to coincide with the direction $\mathbf{t}$. Among all possible transformations (corresponding to an arbitrary rotation around the Z-axis) we select the one specified by the following unitary matrix, where $\rho^2 = m^2 + n^2$:

$$M = \begin{pmatrix} \rho & -\frac{lm}{\rho} & -\frac{nl}{\rho} \\ 0 & \frac{n}{\rho} & -\frac{m}{\rho} \\ l & m & n \end{pmatrix}, \tag{1}$$

Let (X_i, Y_i, Z_i) be the coordinates of p_i in system (X, Y, Z). The orthogonal projection q_i of p_i on π in system (X, Y, Z) in terms of (x_i, y_i, z_i) has coordinates:

$$\left(\rho x_i - \frac{lm}{\rho} y_i - \frac{nl}{\rho} z_i, \frac{n}{\rho} y_i - \frac{m}{\rho} z_i, 0\right) \tag{2}$$

where the last coordinates is 0 as expected.

The points p_1, p_2, p_3, p_4 belong to a circular cylinder of direction $\mathbf{t}$ if and only if the points q_1, q_2, q_3, q_4 are cocircular in π, that is

$$\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \\ X_1^2 + Y_1^2 & X_2^2 + Y_2^2 & X_3^2 + Y_3^2 & X_4^2 + Y_4^2 \end{vmatrix} = 0 \tag{3}$$

Specialising (2) at p_1 and p_2, we obtain that the coordinates of q_1 and q_2 are

$$X_1 = Y_1 = Z_1 = 0; \ X_2 = \rho x_2; \ Y_2 = Z_2 = 0;$$

Since q_i $(i = 3, 4)$ is orthogonal to the unit vector $\mathbf{t}$, by Pythagoras' theorem applied to the lengths of vectors p_i and q_i, we obtain:

$$X_i^2 + Y_i^2 = |q_i|^2 = |p_i|^2 - (\mathbf{t} \cdot p_i)^2$$

Developing determinant $\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n)$ we get

$$\begin{aligned}
&\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n)\\
&= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & \rho x_2 & \rho x_3 - \frac{lm}{\rho} y_3 & \rho x_4 - \frac{lm}{\rho} y_4 - \frac{nl}{\rho} z_4 \\ 0 & 0 & \frac{n}{\rho} y_3 & \frac{n}{\rho} y_4 - \frac{m}{\rho} z_4 \\ 0 & \rho^2 x_2^2 & |p_3|^2 - (\mathbf{t}\cdot p_3)^2 & |p_4|^2 - (\mathbf{t}\cdot p_4)^2 \end{vmatrix}\\
&= \rho x_2 \begin{vmatrix} \frac{n}{\rho} y_3 & \frac{n}{\rho} y_4 - \frac{m}{\rho} z_4 \\ |p_3|^2 - (\mathbf{t}\cdot p_3)^2 & |p_4|^2 - (\mathbf{t}\cdot p_4)^2 \end{vmatrix}\\
&\qquad + \rho^2 x_2^2 \begin{vmatrix} \rho x_3 - \frac{lm}{\rho} y_3 & \rho x_4 - \frac{lm}{\rho} y_4 - \frac{nl}{\rho} z_4 \\ \frac{n}{\rho} y_3 & \frac{n}{\rho} y_4 - \frac{m}{\rho} z_4 \end{vmatrix}
\end{aligned}$$

The first term can be rewritten as

$$\begin{aligned}
& x_2 \begin{vmatrix} n y_3 & n y_4 - m z_4 \\ |p_3|^2 - (\mathbf{t}\cdot p_3)^2 & |p_4|^2 - (\mathbf{t}\cdot p_4)^2 \end{vmatrix}\\
&= -x_2 \begin{vmatrix} \begin{vmatrix} m & y_3 \\ n & 0 \end{vmatrix} & \begin{vmatrix} m & y_4 \\ n & z_4 \end{vmatrix} \\ |p_3|^2 - (\mathbf{t}\cdot p_3)^2 & |p_4|^2 - (\mathbf{t}\cdot p_4)^2 \end{vmatrix}\\
&= -x_2 \begin{vmatrix} m & y_3 & y_4 \\ n & 0 & z_4 \\ 0 & |p_3|^2 - (\mathbf{t}\cdot p_3)^2 & |p_4|^2 - (\mathbf{t}\cdot p_4)^2 \end{vmatrix}\\
&= -x_2 \begin{vmatrix} m & y_3 & y_4 \\ n & 0 & z_4 \\ 0 & (\mathbf{t}\cdot\mathbf{t})\,|p_3|^2 - (\mathbf{t}\cdot p_3)^2 & (\mathbf{t}\cdot\mathbf{t})\,|p_4|^2 - (\mathbf{t}\cdot p_4)^2 \end{vmatrix}
\end{aligned}$$

The second term is

$$\begin{aligned}
& x_2^2 \begin{vmatrix} \rho^2 x_3 - lm y_3 & \rho^2 x_4 - lm y_4 - nl z_4 \\ n y_3 & n y_4 - m z_4 \end{vmatrix}\\
&= x_2^2 \left((1-l^2)x_3 - lm y_3\right)(n y_4 - m z_4)\\
&\qquad -x_2^2 \left((1-l^2)x_4 - lm y_4 - ln z_4\right) n y_3\\
&= x_2^2(n x_3 y_4(1-l^2) - m x_3 z_4(1-l^2) - n x_4 y_3(1-l^2) + l y^3 z^4(n^2+m^2))\\
&= x_2^2(n^2+m^2)(n x_3 y_4 - m x_3 z_4 - n x_4 y_3 + l y_3 z_4)\\
&= x_2^2(n^2+m^2) \begin{vmatrix} l & x_3 & x_4 \\ m & y_3 & y_4 \\ n & 0 & z_4 \end{vmatrix}
\end{aligned}$$

We conclude that

$$\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = x_2^2(m^2+n^2) \begin{vmatrix} l & x_3 & x_4 \\ m & y_3 & y_4 \\ n & 0 & z_4 \end{vmatrix}$$

$$-x_2 \begin{vmatrix} m & y_3 & y_4 \\ n & 0 & z_4 \\ 0 & (\mathbf{t}\cdot\mathbf{t})\,|p_3|^2 - (\mathbf{t}\cdot p_3)^2 & (\mathbf{t}\cdot\mathbf{t})\,|p_4|^2 - (\mathbf{t}\cdot p_4)^2 \end{vmatrix} \tag{4}$$

The preceding analysis, starting from the definition of matrix M and ending with the above expression of $\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n)$ rests on the identity $l^2+m^2+n^2=1$. Thus if we view $\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n)=0$ as the equation of a surface in three-dimensional space, we are actually considering its intersection with the unit sphere of equation $l^2+m^2+n^2=1$. Alternatively, we may view (l,m,n) as a point of the projective space $\mathbb{P}^2$ of non-zero directions, and we conclude that the set of directions for which p_1, p_2, p_3 and p_4 are cocircular verifies $\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n)=0$, which describes a curve of the projective space $\mathbb{P}^2$ of non-zero directions.

When we choose for (l,m,n) one of the directions (p_i,p_j), $1\le i\ne j\le 4$, points p_i and p_j project to the same point in π and correspondingly polynomial $\mathcal{C}_{p_1,p_2,p_3,p_4}$ vanishes. Thus we have the following result.

Theorem 1 *The directions such that four points are cocircular when projecting onto a plane perpendicular to that direction belong to a curve of degree* 3 *of the projective space* $\mathbb{P}^2$*, passing by the* 6 *directions* (p_i,p_j) *for* $1\le i\ne j\le 4$.

4 Cylinders through five points

Let p_1,p_2,p_3,p_4 and p_5 be five distinct points in three dimensional space. We are seeking the direction along which the five points belong to the same circular cylinder. Necessarily, such a direction (l,m,n) must verify the following conditions

$$\begin{array}{l} \mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n)=0 \\ \mathcal{C}_{p_1,p_2,p_3,p_5}(l,m,n)=0 \end{array} \tag{5}$$

More specifically:

Theorem 2 *The number of circular cylinders through five points is at most* 6*. This bound on the number of real solutions is attained by some configurations.*

Proof. Since both equations (4) are of degree 3, by Bézout's theorem the number of points common to the two curves (when finite) is bounded by 9. But the 3 directions p_1p_2, p_1p_3 and p_2p_3 although solutions of the system, do not correspond to directions where the five points are cocircular. Indeed, they identify pairs of circular cylinders sharing one generatrix.

Removal of these three solutions leaves a maximum number of six solutions.

This bound is tight as shown by the example of two regular tetrahedra sharing a face given in Section 2.2, for which there are six feasible directions listed below:

l	m	n
$1/\sqrt{10}$	$1/10\,\sqrt{2}\sqrt{15}$	$1/5\,\sqrt{15}$
$1/\sqrt{10}$	$1/10\,\sqrt{2}\sqrt{15}$	$-1/5\,\sqrt{15}$
$1/\sqrt{10}$	$-1/10\,\sqrt{2}\sqrt{15}$	$1/5\,\sqrt{15}$
$1/\sqrt{10}$	$-1/10\,\sqrt{2}\sqrt{15}$	$-1/5\,\sqrt{15}$
$2/\sqrt{10}$	0	$1/5\,\sqrt{15}$
$2/\sqrt{10}$	0	$-1/5\,\sqrt{15}$

□

Note that this bound is not attained in general if we restrict ourselves to real solutions. To illustrate this remark, it suffices to take as the set of points the vertices of a tetrahedron and a fifth point inside that tetrahedron; in this case, obviously there is no (real) cylinder passing through the five points and the 6 solutions of our system are all with complex coordinates.

If we count, however, the number of complex roots with their multiplicities (which we may have to consider for instance if we are in the neighbourhood of a singular configuration), this bound is exact for almost all set of 5 points, according to Bézout theorem.

5 Cylindrical shells through six points

For metrology applications, we are interested in *zone cylinders* or *cylindrical shells* which are in fact pairs of coaxial cylinders. We call the positive difference of their radii the width of the zone cylinder. Finding a minimal-width zone cylinder containing a set of points P is an important problem in metrology. Such a cylinder passes through (is defined by) six points of P. The partitions of the defining points p_1, p_2, p_3, p_4, p_5 and p_6 between the internal and external cylinder may be of the following types:

— $\{p_1, p_2, p_3, p_4, p_5\}$, $\{p_6\}$ corresponding to the following (cocircularity) conditions:

$$\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = 0$$
$$\mathcal{C}_{p_1,p_2,p_3,p_5}(l,m,n) = 0$$

— $\{p_1, p_2, p_3, p_4\}$, $\{p_5, p_6\}$ corresponding to the following conditions:

$$\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = 0$$
$$\mathcal{C}_{p_1,p_2,p_3,p_5}(l,m,n) = \mathcal{C}_{p_1,p_2,p_3,p_6}(l,m,n)$$

Where the latter specifies that p_5 and p_6 are equidistant from the axis.

— $\{p_1, p_2, p_3\}$, $\{p_4, p_5, p_6\}$ corresponding to the conditions:

$$\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = \mathcal{C}_{p_1,p_2,p_3,p_5}(l,m,n)$$
$$\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = \mathcal{C}_{p_1,p_2,p_3,p_6}(l,m,n)$$

where both conditions specify that $\{p_4,\, p_5,\, p_6\}$ are equidistant from the axis.

In all of these three cases we get a system of two equations of degree 3, and in all cases p_1p_2, p_1p_3 and p_2p_3 are irrelevant solutions, so that in all cases there are at most 6 solutions for any given partition. Partitions of the first, second, and third types are selectable in 6, 15, and 10 ways, respectively, and we have the following theorem:

Theorem 3 *The number of cylindrical shells through six points is at most 186.*

If we are interested in shells of locally minimum width, each cylinder (internal or external) must pass by at least 2 points [2, 8], thus reducing the number of possible partitions of the six points, and the bound on the number of shells to 150.

6 Pair of parallel cylinders through two sets of four points

Given two sets of points, p_1, p_2, p_3, p_4 and p'_1, p'_2, p'_3, p'_4 in three-dimensional space, we are seeking directions along which both sets of points have cocircular projections, or, in other words, directions such that there exists a pair of parallel cylinders in that direction, each of them passing through a set of four points.

Theorem 4 *The number of pairs of parallel circular cylinders through two sets of four points is at most* 9*. This bound on the number of real solutions is attained by some configurations.*

Proof. Clearly the searched directions satisfy the two cubic equations:

$$\begin{aligned} \mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = 0 \\ \mathcal{C}_{p'_1,p'_2,p'_3,p'_4}(l,m,n) = 0 \end{aligned} \tag{6}$$

This system as 9 solutions in $\mathbb{P}^2$ by Bézout theorem. This bound is tight as shown by the example of two regular tetrahedra given in Section 2.3 where the number of different directions attain the bound of 9. □

7 Cylinders through four points of a given radius

We consider now the problem of cylinders with fixed radius through 4 points, which occurs in collision detection or visibility tests.

Theorem 5 *The number of circular cylinders of given radius through four points in general position is at most* 12*. This bound on the number of real solutions is attained in some configurations.*

Proof. The cylinders of radius ρ through four points $p_1, \ldots, p_4$ are determined by the two equations

$$\begin{aligned} \mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) &= 0 \\ radius_{p_1,p_2,p_3}(l,m,n) &= \rho \end{aligned} \tag{7}$$

where $radius_{p_1,p_2,p_3}(l,m,n)$ is the radius of the circle defined by the projections q_1, q_2, q_3 of points p_1, p_2, p_3 on a plane normal to direction (l,m,n). It is well known that the radius R of a circle through points (A, B, C) in the plane is given by the formula

$$\Gamma = R^2 = \frac{d(A,B)^2 \, d(A,C)^2 \, d(B,C)^2}{16\,(area(A,B,C))^2},$$

where $d(A,B)$ is the Euclidean distance between points A and B and $area(A,B,C)$ is the area of the triangle defined by these three points. In our case (that is, $p_1 = (0,0,0)$, $p_2 = (x_2, 0, 0)$, $p_3 = (x_3, y_3, 0)$),) we obtain for the numerator

$$N(l,m,n) = \left(|\mathbf{t}|^2\,|p_1p_2|^2 - (\mathbf{t}\cdot p_1p_2)^2\right)\,\left(|\mathbf{t}|^2\,|p_1p_3|^2 - (\mathbf{t}\cdot p_1p_3)^2\right)\,\left(|\mathbf{t}|^2\,|p_2p_3|^2 - (\mathbf{t}\cdot p_2p_3))^2\right)$$

and for the denominator,

$$D(l,m,n) = 4\begin{vmatrix} 1 & 1 & 1 \\ 0 & \rho\, x_2 & \rho x_3 - \frac{lm}{\rho} y_3 \\ 0 & 0 & \frac{n}{\rho} y_3 \end{vmatrix}^2 |\mathbf{t}|^4 = \left(2\,n\,x_2\,y_3\,|\mathbf{t}|^2\right)^2$$

Thus, system (7) consists of two homogeneous polynomials of degree respectively 3 and 6. By Bézout theorem, the number of projective roots counted with multiplicity, is $3 \times 6 = 18$.

We now observe that each of the directions p_1p_2, p_1p_3 and p_2p_3 satisfies equation $\mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = 0$ (with multiplicity 1 for non collinear points). Moreover, we claim that each of these directions also satisfies equations $N(l,m,n) = 0$ and $D(l,m,n) = 0$. Indeed, setting $\mathbf{t} = p_ip_j$ nullifies one of the three factors of $N(l,m,n)$ (referring to the expression of $N(l,m,n)$ given above, we note that this happens with multiplicity 2). In addition, $\mathbf{t} = p_ip_j$ implies that $\mathbf{t}$ belongs to the plane of the triangle defined by points p_1, p_2, p_3, so that $area(q_1, q_2, q_3) = 0$. In each of these cases the ratio N/D is of the form $0/0$, so that these three degenerate solutions must be rejected and $\mathbf{t} = p_ip_j$ is a solution of multiplicity 2 of $N - \rho^2\, D = 0$. Therefore, the number 18 of projective solutions of system (7) must be reduced by $3 \times 2 = 6$ (three solutions each of multiplicity 2), yielding a bound of 12.

This bound is tight as shown by the example of the regular tetrahedron given in Section 2.1. Indeed in Figure 1, there are 12 segments between a *square*-point (corresponding to a minimum of the radius) and a *triangle*-point (corresponding to a maximum of the radius). Any value of the radius between the maximum and the minimum yields one real solution point on each of these curve segments. □

Remark. It should not be construed, however, that a regular tetrahedron is the only configuration of points attaining the bound stated in the preceding theorem. In fact, it can be easily shown that the set of points $(0,0,0), (2,0,0), (1,a,1), (1.a.-1)$, which gives a regular tetrahedron for $a = \sqrt{2}$, has a cocircularity curve consisting of three great circles, and each of them contains two maxima and two minima in each hemisphere.

8 Cylinders of extremal radius through four points

Theorem 6 *The number of circular cylinders through four points with locally extremal radius is at most* 18*. This bound on the number of real solutions is attained in some configurations.*

Proof. Consider four points $p_1, \ldots, p_4$ in three-dimensional space, where, as usual, and without loss of generality, we assume $p_1 = (0,0,0)$, $p_2 = (x_2, 0, 0)$ and $p_3 = (x_3, y_3, 0)$. Next, we consider the set of cylinders through these four points of square radius Γ. They are defined by the three equations

$$\begin{array}{c} \mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = 0 \\ radius_{p_1,p_2,p_3}(l,m,n)^2 - \Gamma = 0 \\ l^2 + m^2 + n^2 = 1, \end{array} \tag{8}$$

which are homogeneous in the variables (l, m, n, Γ). Using the condition $l^2 + m^2 + n^2 = 1$, the second of these equation can be rewritten as

$$\begin{aligned} \Gamma &= \frac{N(l,m,n)}{D(l,m,n)} \\ &= \frac{\left(|\mathbf{t}|^2\,|p_1p_2|^2 - (\mathbf{t}\cdot p_1p_2)^2\right)\left(|\mathbf{t}|^2\,|p_1p_3|^2 - (\mathbf{t}\cdot p_1p_3)^2\right)\left(|\mathbf{t}|^2\,|p_2p_3|^2 - (\mathbf{t}\cdot p_2p_3)^2\right)}{4y_3^2n^2} \end{aligned}$$

where, with our choices of points, $N(l, m, n)$ has degree 6 and $D(l, m, n) = D(n)$ has degree 2.

Our task is the extremization of Γ subject to the constraints

$$\begin{array}{c} \mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) = 0, \\ l^2 + m^2 + n^2 = 1. \end{array} \tag{9}$$

Introducing Lagrange multipliers λ_1 and λ_2, this reduces to finding the extrema of the function

$$\phi = \frac{N(l,m,n)}{D(n)} + \lambda_1 \mathcal{C}_{p_1,p_2,p_3,p_4}(l,m,n) + \lambda_2(l^2 + m^2 + n^2 - 1) \tag{10}$$

If we now equate to 0 the partial derivatives of ϕ with respect to l, m and n, we can eliminate the multipliers λ_1 and λ_2 from the resulting three equations and obtain a single equation in l, m and n. After some straightforward simplifications and using the abbreviated notation ∂_u for $\frac{\partial}{\partial u}$ we obtain the equation:

$$\Delta := \begin{vmatrix} N\partial_l D - D\partial_l N & N\partial_m D - D\partial_m N & N\partial_n D - D\partial_n N \\ \partial_l C & \partial_m C & \partial_n C \\ l & m & n \end{vmatrix} = 0 \tag{11}$$

to be paired with the equation

$$C(l, m, n) = 0. \tag{12}$$

Since N, D and C are of respective degrees $6, 2$, and 3, Equation (11) has degree 10. By Bézout theorem, the number of joint solutions of system (11)(12) is $10 \times 3 = 30$.

As noted in the preceding section, however, the directions p_1p_2, p_1p_3, p_2p_3 are roots of the polynomials $N(l, m, n), D(n)$, with multiplicity 2 for each of them. Therefore, they are roots of the partial derivatives with multiplicity 1; consequently, they are also roots of all terms of the first line of the determinant, as well as of Equation (11) itself, with multiplicity at least 3.

In fact, we shall now prove algebraically the following stronger result:

Proposition 7 *In Equation (11) the multiplicity of roots p_1p_2, p_1p_3, and p_2p_3 is at least 4.*

Proof of proposition: We shall prove the result for $p_1p_2 = (x_2, 0, 0)$. By analogy the result holds for p_1p_3 and p_2p_3 as well. We shall show that the Taylor expansion of the determinant at point p_1p_2 is expressible as a polynomial of *valuation* 4, i.e., a polynomial all of whose monomials have degree at least 4: this will establish the desired multiplicity of the root. To obtain the expansion at p_1p_2, let $\mathbf{t}' = (l', m, n) = \mathbf{t} - p_1p_2$ and let H_i be a *generic* polynomial in the variables l', m, n of valuation i.

We now observe that for any vector $\mathbf{u}$,

$$\begin{aligned} |\mathbf{t}|^2\ |\mathbf{u}|^2 - (\mathbf{t}\cdot\mathbf{u})^2 &= ((\mathbf{t}' + p_1p_2)\cdot(\mathbf{t}' + p_1p_2))|\mathbf{u}|^2 - (\mathbf{t}'\cdot\mathbf{u} + p_1p_2\cdot\mathbf{u})^2 \\ &= |p_1p_2|^2|\mathbf{u}|^2 - (p_1p_2\cdot\mathbf{u})^2 + H_1 \end{aligned}$$

which specialises as

$$\begin{aligned} |\mathbf{t}|^2\,|p_1p_2|^2 - (\mathbf{t}\cdot p_1p_2)^2 &= \left((x_2+l')^2 + m^2 + n^2\right)x_2^2 - \left((x_2+l')x_2\right)^2 = (m^2+n^2)x_2^2 \\ |\mathbf{t}|^2\,|p_1p_3|^2 - (\mathbf{t}\cdot p_1p_3)^2 &= |p_1p_2|^2\,|p_1p_3|^2 - (p_1p_2\cdot p_1p_3)^2 + H_1 \\ &= x_2^2\,(x_3^2+y_3^2) - (x_2x_3)^2 + H_1 = x_2^2y_3^2 + H_1 \\ |\mathbf{t}|^2\,|p_2p_3|^2 - (\mathbf{t}\cdot p_2p_3)^2 &= |p_1p_2|^2\,|p_2p_3|^2 - (p_1p_2\cdot p_2p_3)^2 + H_1 \\ &= x_2^2\left((x_3-x_2)^2 + y_3^2\right) - (x_2(x_3-x_2))^2 + H_1 = x_2^2y_3^2 + H_1 \end{aligned}$$

Therefore we obtain

$$\begin{aligned} N(x_2+l', m, n) &= \left(|\mathbf{t}|^2\,|p_1p_2|^2 - (\mathbf{t}\cdot p_1p_2)^2\right)\left(|\mathbf{t}|^2\,|p_1p_3|^2 - (\mathbf{t}\cdot p_1p_3)^2\right)\left(|\mathbf{t}|^2\,|p_2p_3|^2 - (\mathbf{t}\cdot p_2p_3)^2\right) \\ &= (m^2+n^2)x_2^2(x_2^2y_3^2 + H_1)(x_2^2y_3^2 + H_1) \\ &= x_2^6y_3^4(m^2+n^2) + H_3 \end{aligned}$$

The entries of the first line of the determinant can be rewritten as

$$\begin{aligned} N\partial_l D - D\partial_l N &= -D\partial_l N = H_4 \\ N\partial_m D - D\partial_m N &= -D\partial_m N = -4y_3^2n^2\ x_2^6y_3^4\ 2m + H_4 = -8x_2^6\ y_3^6mn^2 + H_4 \\ N\partial_n D - D\partial_n N &= x_2^6y_3^4(m^2+n^2)8y_3^2n - 4y_3^2n^2x_2^6y_3^4\ 2n + H_4 \\ &= 8x_2^6\ y_3^6m^2n + H_4 \end{aligned}$$

and correspondingly if we develop the determinant in Equation (11) accordingly to the first row and column we obtain:

$$\begin{vmatrix} H_4 & -8x_2^6\,y_3^6mn^2+H_4 & 8x_2^6\,y_3^6m^2n+H_4 \\ \partial_l C & \partial_m C & \partial_n C \\ l'+x_2 & m & n \end{vmatrix}$$

$$\begin{aligned} &= 8x_2^6\,y_3^6 \begin{vmatrix} 0 & -mn^2 & m^2n \\ 0 & \partial_m C & \partial_n C \\ x_2 & m & n \end{vmatrix} + H_4\partial_l C + H_5 \\ &= -8x_2^7\,y_3^6(\partial_m Cm + \partial_n Cn)mn + H_4 \end{aligned}$$

We now claim that a polynomial H_2 of valuation 2, can be chosen so that

$$C_{p_1,p_2,p_3,p_4}(x_2+l',m,n) = \partial_m Cm + \partial_n Cn + H_2$$

Indeed, from the expression of C in Equation (4) we get:

$$\begin{aligned} &C_{p_1,p_2,p_3,p_4}(x_2+l',m,n) \\ &= H_2 - x_2 \begin{vmatrix} m & y_3 & y_4 \\ n & 0 & z_4 \\ 0 & x_2^2y_3^2+H_1 & x_2^2(y_4^2+z_4^2)+H_1 \end{vmatrix} = \lambda\, m + \mu\, n + H_2 \end{aligned}$$

so that $\partial_m Cm + \partial_n Cn = \lambda\, m + \mu\, n + H_2$. We therefore conclude that the determinant in Equation (11) can be expressed as

$$\begin{aligned} &\Delta + 8x_2^7\,y_3^6C(l,m,n)m\,n \\ &= -8x_2^7\,y_3^6(\partial_m Cm + \partial_n Cn)mn + 8x_2^7\,y_3^6(\partial_m Cm + \partial_n Cn)mn + H_4 \\ &= H_4. \end{aligned}$$

This proves that the multiplicity of p_1p_2 at the ideal $(D,C) = (D + 8x_2^7\,y_3^6C(l,m,n)m\,n, C)$ is at least 4. By symmetry, this also holds for the other two directions.

Thus, the multiplicity of p_1p_2, p_1p_3, and p_2p_3 is at least 4. This result however does not extend to p_1p_4, p_2p_4 or p_3p_4, despite the equivalent roles that these directions play in our problem. In fact, in our analysis we have been considering directions of cocircularity for which the radius of the circle by p_1, p_2 and p_3 is extremal, thus attributing a different role to point p_4. The directions involving p_4 are roots of Equation (12) but not of Equation (11) (except in special configurations).

We now observe that p_1p_2, p_1p_3, and p_2p_3 with multiplicity 4 are improper solutions of our problem and should therefore be removed from the solution pool. In fact, for each of these directions the ratio N/D is of the form $0/0$, so that there is always a circular cylinder passing also by p_4, with no implication of its having extremal radius. Thus, after removing these improper solutions, we conclude that the number of cylinders with extremal radius is at most $30 - 3 * 4 = 18$. [1] To complete the proof, we report below the set of extremal cylinders obtained when the four points are the vertices of a regular tetrahedron. Solving the system of Equations (11) and (12) for the points given in Figure 1 yields 9 distinct solutions with multiplicities (the sum of the multiplicity is 30, that is, there is no complex solutions in this case). In the following table, μ is the multiplicity of the solution, ρ the corresponding radius and by $\mathbf{u} \times \mathbf{v}$ we denote, as customary, the external product of vector $\mathbf{u}$ and $\mathbf{v}$:

$\mathbf{t}$	l	m	n	μ	ρ
p_1p_2	1	0	0	5	$\frac{3}{8}\sqrt{2}$
p_1p_3	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	0	5	$\frac{3}{8}\sqrt{2}$
p_2p_3	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	0	5	$\frac{3}{8}\sqrt{2}$
p_1p_4	$-\frac{1}{2}$	$\frac{1}{6}\sqrt{3}$	$\frac{1}{3}\sqrt{6}$	1	$\frac{3}{8}\sqrt{2}$
p_2p_4	$\frac{1}{2}$	$\frac{1}{6}\sqrt{3}$	$\frac{1}{3}\sqrt{6}$	1	$\frac{3}{8}\sqrt{2}$
p_3p_4	0	$-\frac{1}{3}\sqrt{3}$	$\frac{1}{3}\sqrt{6}$	1	$\frac{3}{8}\sqrt{2}$
$p_1p_2 \times p_4p_3$	0	$\frac{1}{3}\sqrt{6}$	$\frac{1}{3}\sqrt{3}$	4	$1/2$
$p_1p_4 \times p_3p_2$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{3}}$	4	$1/2$
$p_1p_3 \times p_2p_4$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	4	$1/2$

In accordance with the preceding argument, directions p_1p_2, p_1p_3 and p_2p_3 are improper solutions of the system with multiplicity 4; their removal (with multiplicity 4) leaves each of them with multiplicity, thereby revealing their expected equivalence with directions p_1p_4, p_2p_4 and p_3p_4, so that the six edges correspond to solutions of extremal radius (maximal as observed in Section 2.1). On the other hand, directions $p_1p_2 \times p_4p_3$, $p_1p_4 \times p_3p_2$ and $p_1p_3 \times p_2p_4$ (linking the mid-points of two opposite edges) are solutions of the system with multiplicity 4 and correspond to minima of the radius. In the example of the regular tetrahedron we get 18 real solutions counted with their multiplicity, matching the upper bound. □

The previous configuration (a regular tetrahedron) is deformed by slightly perturbing the positions of the points $p_1, \ldots, p_4$ with the following results:

- The first three solution of multiplicity 5 give rise to a point (of multiplicity 1) corresponding to a local maximum and to the irrelevant solutions p_1p_2, p_1p_3, p_2p_3, each with multiplicity 4.
- The next three simple points remain simple and correspond to cylinders with locally maximal radius.
- Each point of the last set of three of multiplicity 4 is transformed into two distinct real minima and two additional complex points.

[1]This does not prevent any of the three directions from being a legitimate solution in special configurations.

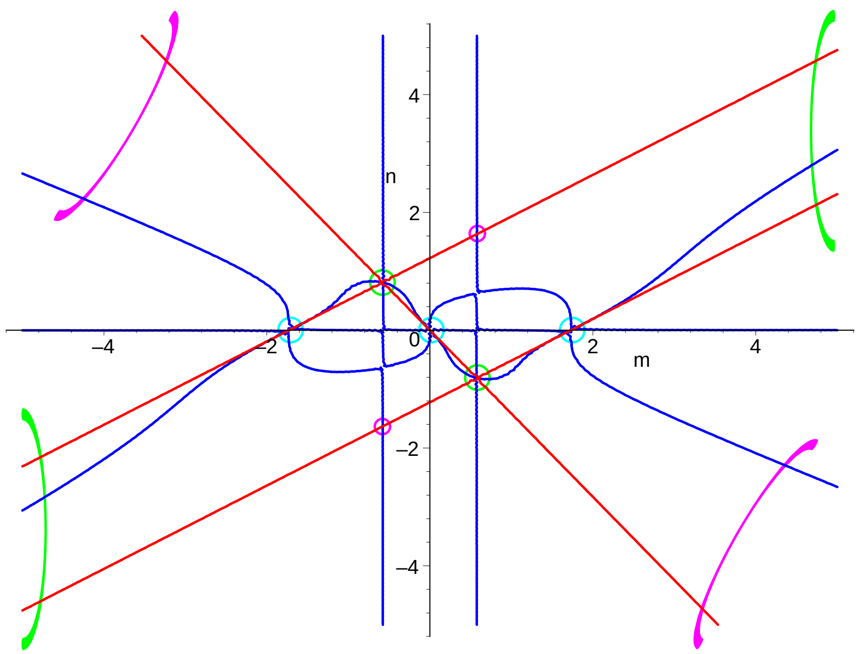

Figure 5: Projective view of the curves defined by the homogeneous system of Equations (11) and (12) for the regular tetrahedron.

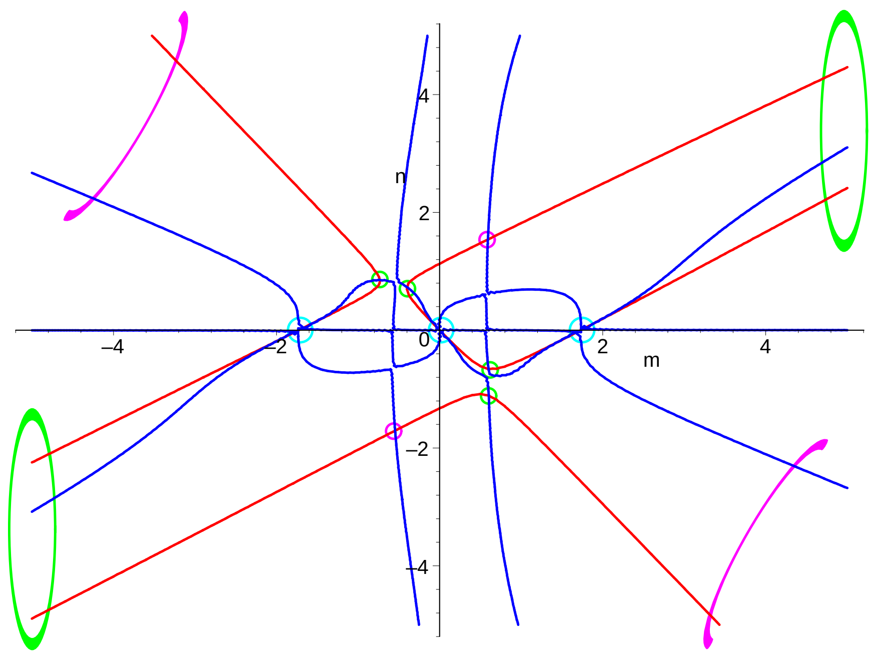

Figure 6: Projective view of the curves defined by the homogeneous system of Equations (11) and (12) for the perturbed regular tetrahedron.

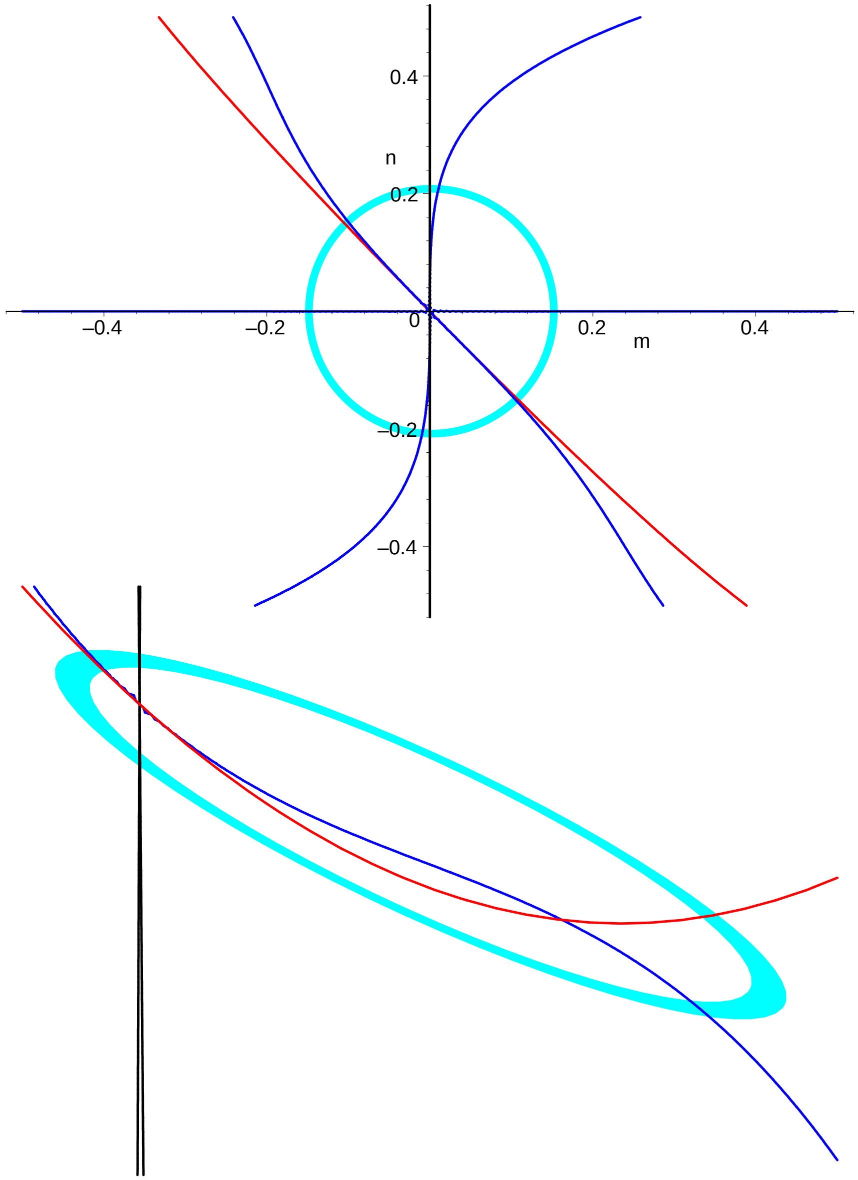

Figure 7: Blow-up of the projective view of the solution of the homogeneous system of Equations (11) and (12) for the perturbed regular tetrahedron in the neighborhood of the direction p_1p_2.

Equations (11) and (12) define two surfaces in 3D space l, m, n. In Figures 5 and 6 we present a cross section of these two surfaces with the plane $l = 1$ in the case of the regular tetrahedron (Figure 5) and the perturbed regular tetrahedron (Figure 6).

The red curve corresponds to Equation (12) (co-circularity) and consists of three straight lines in the case of the regular tetrahedron (as already noted in Figure 2). The blue curve corresponds to Equation (11) (extremality). The solutions p_1p_4, p_2p_4 and p_3p_4, circled in pink in the two figures, appear in Figure 5 as non tangent intersections between blue and red curves (one of the three points, p_3p_4, is at infinity); these solutions persist in Figure 6. The solutions $p_1p_2 \times p_4p_3$, $p_1p_4 \times p_3p_2$ and $p_1p_3 \times p_2p_4$ appear circled in green in Figure 5 at the intersections between pairs of branches of the red curve (one of the three points is at infinity). Figure 6 clearly shows that, when the tetrahedron is perturbed, each of these points splits into non tangent intersections between red and blue curves. Finally the solutions p_1p_2, p_1p_3 and p_2p_3 as tangencies of multiplicity 5 between red and blue curves on the axis $n = 0$ are circled in light blue in Figure 5; in the perturbed version (Figure 6) each of these points splits into one tangent contact between the two curves and one non tangent intersection. The situation, which is not very discernible in Figure 6, is more clearly illustrated in Figure 7, where the circular domain around the origin (corresponding to solution p_1p_2) is transformed by an appropriate affine transformation into and elliptical domain. This blow-up gives evidence to the existence of a tangent intersection and a simple intersection.

Theorem 8 *The number of circular cylinders through four points of locally minimal (resp. maximal) radius is at most 9.*

Proof. This theorem is an easy corollary of Theorem 6. Indeed, the set of directions such that the four points are cocircular is a degree 3 curve of the projective space $\mathbb{P}^2$ (Theorem 1). This curve can be decomposed into the union of some closed curves (three projective lines in the case of the regular tetrahedron of Figure 1). To each point of these curves there corresponds a unique value of the radius of a cylinder, and, when tracing any of these curves, we visit minima and maxima of radius in identical numbers. Since the total number of extrema, with multiplicity, is at most 18, the total number of minima or maxima is at most 9. $\square$

Referring to the example of the (perturbed) regular tetrahedron, this bound does not appear to be tight. Indeed, we conjecture that the exact bound on the number of real maxima (or minima) is 6.

References

[1] Pankaj K. Agarwal, Boris Aronov, and Micha Sharir. Line traversals of balls and smallest enclosing cylinders in three dimensions. In *Proc. 8th ACM-SIAM Sympos. Discrete Algorithms*, pages 483–492, 1997.

[2] Pankaj K. Agarwal, Boris Aronov, and Micha Sharir. Exact and approximation algorithms for minimum-width cylindrical shells. In *Proc. 11th ACM-SIAM Sympos. Discrete Algorithms*, pages 510–517, 2000.

[3] E. Anderson, Z. Bai, C. Bischof, J.W. Demmel, J.J. Dongara, J. Du Croz, A. Greenbaum, S. Hammerling, A. McKenney, and D. Sorensen. LAPACK : A portable linear algebra library for high-performance computers. Technical Report CS-90-105, Comp. Sci. Dept, Univ. of Tennessee, Knoxville, (USA), 1990.

[4] Julien Basch, Olivier Devillers, Marc van Kreveld, Monique Teillaud, and Antoine Vigneron. On the number of Delaunay triangulations of the projections of a set of points in three dimensions. Rapport de recherche, INRIA. to appear.

[5] O. Bottema and G.R. VeldKamp. On the lines in space with equal distance to n given points. *Geometrica Dedicata*, 6:121–129, 1977.

[6] R.M. Corless, P.M. Gianni, and B.M. Trager. A reordered Schur factorization method for zero-dimensional polynomial systems with multiple roots. In W.W. Küchlin, editor, *Porc. ISSAC*, pages 133–140, 1997.

[7] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer Verlag, New York, 1992.

[8] Olivier Devillers and Franco P. Preparata. Evaluating the cylindricity of a nominally cylindrical point set. In *Proc. 11th ACM-SIAM Sympos. Discrete Algorithms*, pages 518–527, 2000.

[9] I. Mac Donald, J. Pach, and T. Theobald. Common tangents to four unit balls in R^3. *Discrete Comput. Geom.* to appear.

[10] M. Elkadi and B. Mourrain. Géométrie algébrique effective en dimension 0 : de la théorie à la pratique, 2001. Notes de cours, DEA de Mathématiques, Université de Nice `http://www.inria.fr/saga/mourrain/`.

[11] I.Z. Emiris and B. Mourrain. Matrices in Elimination Theory. *J. of Symbolic Computation*, 28(1&2):3–44, 1999.

[12] D. Lichtblau. Finding cylinders through 5 points in $\mathbb{R}^3$. Submitted.

[13] B. Mourrain. Computing isolated polynomial roots by matrix methods. *J. of Symbolic Computation, Special Issue on Symbolic-Numeric Algebra for Polynomials*, 26(6):715–738, Dec. 1998.

[14] B. Mourrain. A new criterion for normal form algorithms. In M. Fossorier, H. Imai, Shu Lin, and A. Poli, editors, *Proc. AAECC*, volume 1719 of *LNCS*, pages 430–443. Springer, Berlin, 1999.

[15] B. Mourrain and H. Prieto. A framework for symbolic and numeric computations. Rapport de Recherche 4013, INRIA, 2000.

[16] B. Mourrain and Ph. Trébuchet. Solving projective complete intersection faster. pages 231–238. New-York, ACM Press., 2000.

[17] Elmar Schömer, Jurgen Sellen, Marek Teichmann, and Chee Yap. Smallest enclosing cylinders. In *Proc. 12th Annu. ACM Sympos. Comput. Geom.*, pages C13–C14, 1996.

9 Appendix: Actual computation of solutions

We are going to describe two methods, that we have used to solve explicitly the different problems, presented in the previous sections. The first one is a general approach based on eigencomputation, the second one is specific to curves in the plane. General references on this topic can be found in [7, 10]. Code to solve the specific problems on cylinders described in that paper can be found at `http://www-sop.inria.fr/galaad/demo/Cylindre`.

9.1 Solving polynomial equations by eigencomputation

We consider here the general setting of m equations $f_1 = 0, \dots, f_m = 0$ in n variables $t_1, \dots, t_n$, with coefficients in $\mathbb{K} = \mathbb{R}$. The polynomials $f_1, \dots, f_m$ are elements of the polynomial ring denoted by $R = \mathbb{K}[t_1, \dots, t_n]$. They generate the ideal $I = (f_1, \dots, f_m)$. In our examples, we consider two homogeneous equations f_1, f_2 of our problem and the equation $f_3 = l^2 + m^2 + n^2 - 1$. We have $t_1 = l, t_2 = m, t_3 = n$ and $R = \mathbb{R}[l, m, n]$. By considering f_3, we will double the number of solutions and treat the problem in an handle setting instead of a projective one.

The quotient algebra of classes of polynomials modulo the ideal I is denoted by $\mathcal{A}$. Its dual space (that is the set of linear forms from $\mathcal{A}$ to $\mathbb{K}$) is denoted by $\widehat{\mathcal{A}}$. Consider the map of multiplication by a variable t_i in $\mathcal{A}$

$$\begin{array}{rcl} M_i : \mathcal{A} & \rightarrow & \mathcal{A} \\ a & \mapsto & a\, t_i. \end{array}$$

Assume that the number of solutions our system is finite: $\zeta_1, \dots, \zeta_d$ then $\mathcal{A}$ is a finite vector space and we have the following theorem:

Theorem 9 *[13] The common eigenvalues of all the M_i are the i^{th} coordinates $(\zeta_k)_i$ of the roots $\zeta_1, \dots, \zeta_d$. The common eigenvectors of all the M_i^t are the evaluation operators $\mathbf{1}_{\zeta_1}, \dots, \mathbf{1}_{\zeta_m}$ where $\mathbf{1}_{\zeta_i} : p \mapsto p(\zeta_i) \in \widehat{\mathcal{A}}$.*

The dimension of $\mathcal{A}$ will be the number of solutions counted with multiplicity. In our examples, it will be the Bézout bound, that is the product of the degrees of the equations. This theorem shows that solving the polynomial equations reduces to computing the matrix of the operators of multiplication M_i and then to perform an eigenvector/eigenvalue decomposition.

The first step is performed according to the algorithm proposed in [14], and its specialisation to generic projective complete intersection [16] (which is the case that we are considering). We get matrices of size the Bézout bounds. The eigenvector computations is performed in ALP [15] with LAPACK [3] subroutines. We remove from these solutions the redundant one, corresponding to the directions p_1p_2, p_1p_3, p_2p_3.

The numerical computations of the solutions is performed in a C++ implementations in ALP. Here are the timing and the accuracy that we get for the different problems:

Problem	*time*	$max(\|f_i\|)$
cylinder through 5 points	$0.03s$	$5 \cdot 10^{-9}$
parallel cylinders through 2×4 points	$0.03s$	$5 \cdot 10^{-9}$
cylinder through 4 points, extremal radius	$2.9s$	10^{-6}

Computations performed on an Intel PII 400 128 Mo of Ram

The tremendous difference of computation time between the case of Section 8 and the two other and the difference of accuracy should be noticed. The explanation for these differences is in the method of resolution that we used. For the first two we knew already that the roots were of multiplicity one in the generic case, ie each eigenspace was of dimension one, this enabled us to compute the roots just by computing the eigenvectors of only one *generic* M_f. But in the case of extremal radii the trouble was that generically the system had multiplicities and we had to consider another more sophisticated and costly method to recover the roots *and* their multiplicities (see for instance [6] for further details).
Figure 8 shows the six cylinders solutions for the set of points corresponding to the regular tetrahedron (the real solutions are displayed with pov-ray).

9.2 Solving by resultant computation

Another approach for solving a (square) polynomial systems consists in using resultant constructions. We hide a variable (that we consider as a parameter) in order to get a system with one equation more that the number of remaining variables. Then we use a resultant formulation [11], in order to get a condition on this parameter such that the overdetermined system has a solution. From this condition, we deduce the values of the hidden variable for the roots of our system. We recover the values of the other coordinates, by simple linear algebra operations.

In our example, we substitute l by 1, hide the variable m and compute the Sylvester matrix of the polynomials f_1, f_2 with respect to the n. If f_i is of degree d_i, with respect to n, we get a square $(d_1 + d_2) \times (d_1 + d_2)$ matrix $\mathtt{S}(m)$ whose entries are polynomial in m

$$\mathtt{S}(m) = \mathtt{S}_k m^k + \cdots + \mathtt{S}_1 m + \mathtt{S}_0,$$

where $\mathtt{S}_i$ has constant coefficients. We are looking for the values of m such that the Sylvester resultant vanishes ($\det(\mathtt{S}(m)) = 0$) or more precisely to values of (m, n) such that

$$[1, n, \ldots, n^{d_1+d_2-1}]\, \mathtt{S}(m) = 0.$$

This can also be transformed into a generalised eigenproblem:

$$\left(\begin{bmatrix} \mathbf{0} & \mathbb{I} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbb{I} \\ \mathtt{S}_0^t & \mathtt{S}_1^t & \ldots & \mathtt{S}_{k-1}^t \end{bmatrix} - \lambda \begin{bmatrix} \mathbb{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & -\mathtt{S}_k^t \end{bmatrix} \right) \mathbf{w} = 0.$$

The corresponding eigenvalues yields the possible values for m: For each value of these values m_0, we deduce from the corresponding eigenvectors in the kernel of $\mathtt{S}^t(m_0)$.

If the kernel is of dimension 1, it is generated by $[1, n_0, \ldots, n_0^{d_1+d_2-1}]$, where $(1, m_0, n_0)$ is a solution of $f_1 = f_2 = 0$.

If the kernel of $\mathtt{S}^t(m_0)$ is generated by $\mathtt{K} = [\mathtt{k}_1, \ldots, \mathtt{k}_r]$ (with $\mathtt{S}^t(m_0)\mathtt{K} = \mathbf{0}$), we compute the generalised eigenvalues of $\Delta_1 - \lambda\, \Delta_0$ where Δ_0 (resp. Δ_1) is the first $r \times r$ sub-matrix of $\mathtt{K}$ (resp. the $r \times r$ sub-matrix of $\mathtt{K}$ formed by the rows $1, \ldots, r+1$). From these eigenvalues, we deduce the values of the corresponding n_0 *above* m_0. Notice that this does not yield the multiplicity of the intersection point $(1, m_0, n_0)$ of $f_1 = f_2 = 0$.

In order to complete our set of solutions, we also compute the points *at infinity* corresponding to $l = 0$, which are the projective roots of the gcd of the homogeneous polynomials $f_1(0, m, n), f_2(0, m, n)$.

We refer to [11] for more information on resultant techniques for solving polynomial equations.

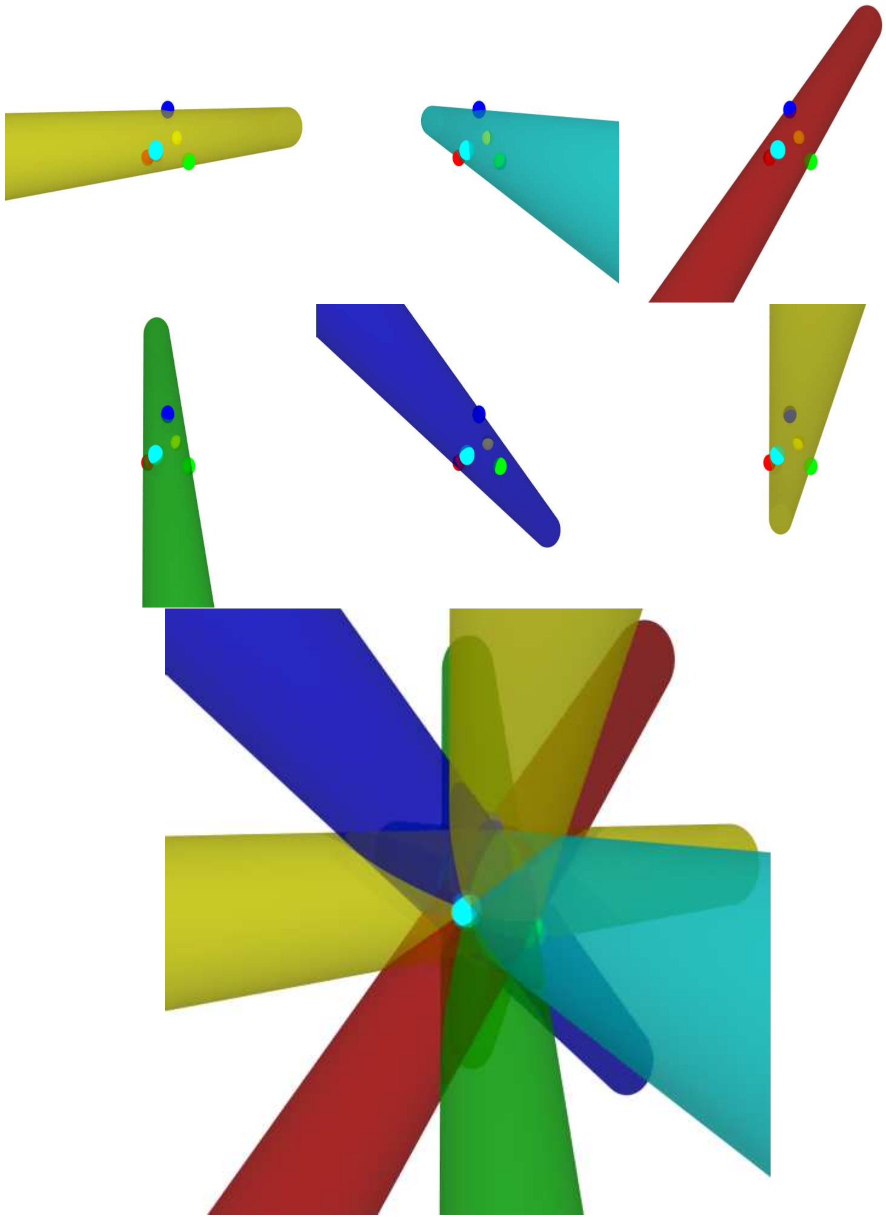

Figure 8: The six cylinders through five points

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - B.P. 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Lorraine : Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 Villers lès Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
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